Preface

This fourth volume in The Theoretical Minimum (TTM) series on general relativity is the natural continuation of the third volume on special relativity.

In special relativity, Einstein, starting from a very simple principle – the laws of physics should be the same in indistinguishable Galilean referentials – deeply clarified in a couple of papers published in 1905 the various disturbing observations physicists had made and the equations they had written in the last years of the nineteenth and the first years of the twentieth century concerning light and other phenomena.

Special relativity led to a strange description of space-time where time and space were inextricably mingled. For instance, it explained how particles whose lifetime is measured in fractions of a second can have, in our referential, a travel time from the Sun to Earth of more than eight minutes.

Then, from 1907 until 1915, essentially alone, Einstein reproduced his feat starting now from another very simple principle – acceleration and uniform gravity are equivalent. He generalized special relativity to a space-time containing massive bodies. The theory is called *general relativity* (GR). It led to an even stranger description of space-time where masses bend light and more generally warp space and time.

In lecture 1, we prepare the groundwork. We show how the equivalence principle inescapably leads to the bending of light rays by massive bodies.

Lecture 2 is devoted to tensor mathematics because in GR we must frequently change referentials and the equations relating coordinates in one referential to coordinates in another are tensor equations.

Then a large part of the theory is expressed using tensor equations because they have the great quality that if they hold in one referential, they hold in all of them.

Lectures 3, 4, and 5 are devoted to the geometry of Riemannian space and Minkowskian space-time because it can be said, very summarily, that gravity is geometry in a Minkowskian space-time.

In lectures 6, 7, and 8, we explore black holes, not so much because they are interesting astronomical phenomena per se, than because they are the equivalent in Minkowskian space-time of point masses in Newtonian mechanics. Space-time however presents a stranger behavior in the vicinity of a black hole than Newtonian space in the vicinity of a point mass. Understanding well black holes, the *metric* they create, their horizon, time and gravity in the vicinity of their horizon, the way people in and out of a black hole can communicate, etc. is a prerequisite to understanding GR.

In lecture 9 we sketch the derivation of Einstein field equations. And in lecture 10 we present a simple application predicting gravity waves.

This book, as the preceding ones in the series, is adapted from a course I gave for several years, with much pleasure, at Stanford in the Continuing Studies program to an audience of adults.

My coauthor this time is André Cabannes. Even though he is not a professional scientist, his scientific training, including a Stanford doctorate and a couple of years of teaching applied mathematics at the Massachusetts Institute of Technology (MIT), helped him assist me.

May Einstein's way of doing physics – starting from the simplest principles and pursuing dauntlessly the mathematics and the physics to their ultimate consequences, however unsettling they may be – as I have strived to show in this book, be a source of inspiration to young and future physicists.

Leonard Susskind
Palo Alto, California
Fall 2022

Ten years ago, when two of my children, then in their late teens, were studying sciences to enter the French system of grandes écoles, I decided to brush up what I had learned in the seventies in order to accompany them in their studies. I discovered that the Internet had profoundly changed the learning landscape. Beside reading books, one could now also take excellent free courses on the Net. I leisurely attended courses in mathematics, physics, computer science, etc. from MIT, Stanford, and other places. The subject matters often were better explained, the courses more lively and easier to understand, than what I had experienced in the past. One could choose courses by the world's best teachers.

Among these courses was The Theoretical Minimum series by Leonard Susskind, famous among other reasons for his pioneering work on string theory. I liked them so much that when I discovered that two of his filmed physics courses had already been transformed into books, I decided to translate them in French. Later I also translated the third book. Then, since the next volume didn't exist in English yet, I took up writing the English notes as well, having in mind that this work might turn out to be useful. After a lot more work with Professor Susskind and Basic Books team, volume 4 in The Theoretical Minimum series, on general relativity, that you hold in your hands is the result.

I belong to the group of people to whom these so-called Continuing Studies courses were intended: individuals who studied physics at the undergraduate and sometimes graduate level when they were students, then did other things in life, but kept an interest in sciences and would like to have some exposure to where physics stands today at a level above plain vulgarization. Indeed, personally, I have always found vulgarization more confusing and harder to understand than real explanations with some equations.

Leonard's courses gave me access to Lagrangian classical mechanics, quantum mechanics, and classical field theory with a clarity that I had never known before. With his pedagogy and presentation it becomes a pleasure to learn. Of course, it is all the more true when there is no examination of any sort at the end. But the courses and books turned out to be useful for students as well, to prepare for more advanced and academic studies.

So whether you are someone who only wants to have some real understanding of what general relativity is about – the stuff on gravitation that is geometry, masses that bend space, light, and time, black holes out there that you should avoid falling into, gravity waves that we begin to detect, etc. – or you are a student in physics who wants to have a first presentation of general relativity, this book is for you.

André Cabannes
Saint-Cyr-sur-mer,
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Fall 2022

Lecture 1: Equivalence Principle and Tensor Analysis

Andy: *So if I am in an elevator and I feel really heavy, I can't know whether the elevator is accelerating or you mischievously put me on Jupiter?*

Lenny: *That's right, you can't.*

Andy: *But, at least on Jupiter, if I keep still, light rays won't bend.*

Lenny: *Oh yes they will.*

Andy: *Hmm, I see.*

Lenny: *And if you are falling into a black hole, beware, things will get really strange. But, don't worry, I'll shed some light on this.*

Andy: *Er, bent or straight?*

Introduction

General Relativity is the fourth volume in The Theoretical Minimum (TTM) series. The first three were devoted respectively to classical mechanics, quantum mechanics, and special relativity and classical field theory. The first volume laid out the Lagrangian and Hamiltonian description of physical phenomena and the principle of least action, which is one of the fundamental principles underlying all of physics (see volume 3, lecture 7 on fundamental principles and gauge invariance). They were used in the first three volumes and will continue in this and subsequent ones.

Physics extensively uses mathematics as its toolbox to construct formal, quantifiable, workable theories of natural phenomena. The main tools we used so far are trigonometry, vector spaces, and calculus, that is, differentiation and integration. They have been explained in volume 1 as well as in brief refresher sections in the other volumes. We assume that the reader is familiar with these mathematical tools and with the physical ideas presented in volumes 1 and 3. The present volume 4, like volumes 1 and 3 (but unlike volume 2), deals with classical physics in the sense that no quantum uncertainty is involved.

We also began to make light use of tensors in volume 3 on special relativity and classical field theory. Now with general relativity we are going to use them extensively. We shall study them in detail. As the reader remembers, tensors generalize vectors. Just as vectors have different representations, with different sets of numbers (components of the vector) depending on the basis used to chart the vector space they form, this is true of tensors as well. The same tensor will have different components in different coordinate systems. The rules to go from one set of components to another will play a fundamental role. Moreover, we will work mostly with *tensor fields*, which are sets of tensors, a different tensor attached to each point of a space. Tensors were invented by Ricci-Curbastro and Levi-Civita[1] to develop work of Gauss[2]

[1]Gregorio Ricci-Curbastro (1853–1925) and his student Tullio Levi-Civita (1873–1941) were Italian mathematicians. Their most important joint paper is "Méthodes de calcul différentiel absolu et leurs applications," in *Mathematische Annalen* 54 (1900), pp. 125–201. They did not use the word *tensor*, which was introduced later by other people.

[2]Carl Friedrich Gauss (1777–1855), German mathematician.

on curvature of surfaces and Riemann[3] on non-Euclidean geometry. Einstein[4] made extensive use of tensors to build his theory of general relativity. He also made important contributions to their usage: the standard notation for indices and the Einstein summation convention.

In *Savants et écrivains* (1910), Poincaré[5] writes that "in mathematical sciences, a good notation has the same philosophical importance as a good classification in natural sciences." In this book we will take care to always use the clearest and lightest notation possible.

Equivalence Principle

Einstein's revolutionary papers of 1905 on special relativity deeply clarified and extended ideas that several other physicists and mathematicians – Lorentz,[6] Poincaré, and others – had been working on for a few years. Einstein investigated the consequences of the fact that the laws of physics, in particular the behavior of light, are the same in different inertial reference frames. He deduced from that a new explanation of the Lorentz transformations, of the relativity of time, of the equivalence of mass and energy, etc.

After 1905, Einstein began to think about extending the principle of relativity to any kind of reference frames, frames that may be accelerating with respect to one another, not just inertial frames. An *inertial frame* is one where Newton's laws, relating forces and motions, have simple expressions. Or, if you prefer a more vivid image, and you know how to juggle, it is a frame of reference in which you can juggle with no problem – for instance in a railway car moving uniformly, without jerks or accelerations of any sort. After ten years of efforts to build a theory extending the principle of relativity to frames with acceleration and taking into account gravitation in a novel way, Einstein published his work in November 1915. Unlike special relativity, which topped off the work of many, general relativity is essentially the work of one man.

[3] Bernhard Riemann (1826–1866), German mathematician.

[4] Albert Einstein (1879–1955), German, Swiss, German again, and finally American physicist.

[5] Henri Poincaré (1854–1912), French mathematician.

[6] Hendrik Antoon Lorentz (1853–1928), Dutch physicist.

We shall start our study of general relativity pretty much where Einstein started. It was a pattern in Einstein's thinking to start with a really simple elementary fact, which almost a child could understand, and deduce these incredibly far-reaching consequences. We think that it is also the best way to teach it, to start with the simplest things and deduce the consequences.

So we shall begin with the *equivalence principle*. What is the equivalence principle? It is the principle that says that *gravity is in some sense the same thing as acceleration*. We shall explain precisely what is meant by that, and give examples of how Einstein used it. From there, we shall ask ourselves: what kind of mathematical structure must a theory have for the equivalence principle to be true? What kind of mathematics must we use to describe it?

Most readers have probably heard that general relativity is a theory not only about gravity but also about geometry. So it is interesting to start at the beginning and ask what is it that led Einstein to say that gravity has something to do with geometry. What does it mean to say that "gravity equals acceleration"? You all know that if you are in an accelerated frame of reference, say, an elevator accelerating upward or downward, you feel an effective gravitational field. Children know this because they feel it.

What follows may be overkill, but making some mathematics out of the motion of an elevator is useful to see in a very simple example how physicists transform a natural phenomenon into mathematics, and then to see how the mathematics is used to make predictions about the phenomenon.

Before proceeding, let's stress that the following study on an elevator, and the laws of physics as perceived inside it, is simple. Yet it is a first presentation of very important concepts. It is fundamental to understand it very well. Indeed, we will often refer to it. In lectures 4 to 9, it will strongly help us understand acceleration, gravitation, and how gravitation "warps" space-time.

So let's imagine the Einstein thought experiment where somebody is in an elevator; see figure 1. In later textbooks, it got promoted to a rocket ship. But I have never been in a rocket ship, whereas

I have been in an elevator. So I know what it feels like when it accelerates or decelerates. Let's say that the elevator is moving upward with a velocity v.

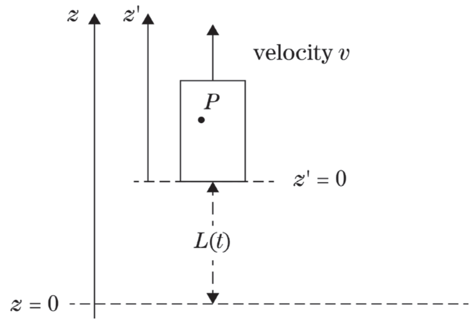

Figure 1: Elevator and two reference frames.

So far the problem is one-dimensional. We are only interested in the vertical direction. There are two reference frames: one is fixed with respect to Earth. It uses the coordinate z. The other is fixed with respect to the elevator. It uses the coordinate z'. A point P anywhere along the vertical axis has two coordinates: coordinate z in the stationary frame, and coordinate z' in the elevator frame. For instance, the floor of the elevator has coordinate $z' = 0$. Its z-coordinate is the distance L, which is obviously a function of time. So we can write for any point P

$$z' = z - L(t) \tag{1}$$

We are going to be interested in the following question: if we know the laws of physics in the frame z, what are they in the frame z'?

One warning about this lecture: at least at the start, we are going to ignore special relativity. This is tantamount to saying that we are pretending that the speed of light is infinite, or that we are talking about motions so slow that the speed of light can be regarded as infinitely fast. You might wonder: if general relativity is the generalization of special relativity, how did Einstein manage to start thinking about general relativity without including special relativity?

The answer is that special relativity has to do with very high velocities, while gravity has to do with heavy masses. There is a range of situations where gravity is important but high velocities are not. So Einstein started out thinking about gravity for slow velocities, and only later combined it with special relativity to think about the combination of fast velocities and gravity. And that became the general theory.

Let's see what we know for slow velocities. Suppose that z' and z are both inertial reference frames. That means, among other things, that they are related by uniform velocity:

$$L(t) = vt \tag{2}$$

We have chosen the coordinates such that when $t = 0$, they line up. At $t = 0$, for any point, z and z' are equal. For instance, at $t = 0$ the elevator's floor has coordinate 0 in both frames. Then the floor starts rising, its height z equaling vt. So for any point we can write equation (1). In view of equation (2), it becomes

$$z' = z - vt \tag{3}$$

Notice that this is a *coordinate transformation* involving space and time. For readers who are familiar with volume 3 of TTM on special relativity, this naturally raises the question: what about time in the reference frame of the elevator? If we are going to forget special relativity, then we can just say that t' and t are the same thing. We don't have to think about Lorentz transformations and their consequences. So the other half of the coordinate transformation would be $t' = t$.

We could also add to the stationary frame a coordinate x going horizontally and a coordinate y jutting out of the page. Correspondingly, coordinates x' and y' could be attached to the elevator; see figure 2. The x-coordinate will play a role in a moment with a light beam. As long as the elevator is not sliding horizontally, x' and x can be taken to be equal. Same for y' and y.

For the sake of clarity of the drawing in figure 2, we offset a bit the elevator to the right of the z-axis. But think of the two vertical axes as actually sliding on each other, and at $t = 0$ the two origins O and O' coincide. Once again, the elevator moves only vertically.

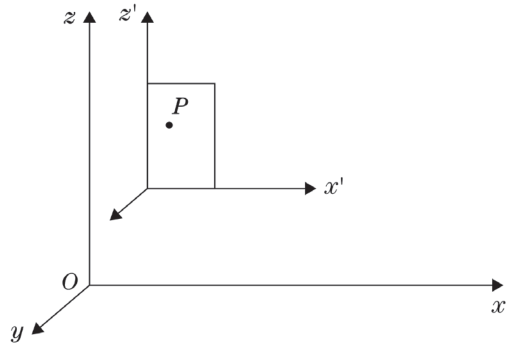

Figure 2: Elevator and two reference frames, three axes in each case.

Finally our complete coordinate transformation is

$$\begin{aligned} z' &= z - vt \\ t' &= t \\ x' &= x \\ y' &= y \end{aligned} \tag{4}$$

It is a coordinate transformation of space-time coordinates. For any point P in space-time, it expresses its coordinates in the moving reference frame of the elevator as functions of its coordinates in the stationary frame. It is rather trivial. Only one coordinate, namely z, is involved in an interesting way.

Let us look at a law of physics expressed in the stationary frame. Take Newton's law of motion $F = ma$ applied to an object or a particle. The acceleration a is $\ddot{z}$, where z is the vertical coordinate of the particle. So we can write

$$F = m\ddot{z} \tag{5}$$

As we know, $\ddot{z}$ is the second time derivative of z with respect to time – it is called the vertical acceleration – and F of course is the vertical component of force. The other components we will take to be zero. Whatever force is exerted, it is exerted vertically. What could this force be due to? It could be related to the elevator or not. There could be some charge in the elevator pushing on the particle. Or it could just be a force due to a rope attached to the ceiling and to the particle that pulls on it. There could be a field force along the vertical axis. Any kind of force could be acting on the particle. Whatever the causes, we know from Newton's law that the equation of motion of the particle, expressed in the original frame of reference, is given by equation (5).

What is the equation of motion expressed in the primed frame? This is very easy. All we have to do is figure out what the original acceleration is in terms of the primed acceleration. What is the primed acceleration? It is the second derivative with respect to time of z'. Using the first equation in equations (4)

$$z' = z - vt$$

one differentiation gives

$$\dot{z'} = \dot{z} - v$$

and a second one gives

$$\ddot{z'} = \ddot{z}$$

The accelerations in the two frames of reference are the same.

All this should be familiar. But I want to formalize it to bring out some points. In particular, I want to stress that *we are doing a coordinate transformation*. We are asking how the laws of physics change in going from one frame to another. What can we now say about Newton's law in the primed frame of reference? We substitute $\ddot{z'}$ for $\ddot{z}$ in equation (5). As they are equal, we get

$$F = m\ddot{z'} \tag{6}$$

We found that Newton's law in the primed frame is exactly the same as Newton's law in the unprimed frame. That is not surprising. The two frames of reference are moving with uniform velocity relative to each other. If one of them is an inertial frame, the other is an inertial frame. Newton taught us that the laws of physics are the same in all inertial frames. It is sometimes called the *Galilean principle of relativity*. We just formalized it.

Let's turn to an accelerated reference frame.

Accelerated Reference Frames

Suppose that $L(t)$ from figure 1 is increasing in an accelerated way. The height of the elevator's floor is now given by

$$L(t) = \frac{1}{2}gt^2 \tag{7}$$

We use the letter g for the acceleration because we will discover that the acceleration mimics a gravitational field – as we feel when we take an elevator and it accelerates. We know from volume 1 of TTM on classical mechanics or from high school, that this is a uniform acceleration. Indeed, if we differentiate $L(t)$ with respect to time, after one differentiation we get

$$\dot{L} = gt$$

which means that the velocity of the elevator increases linearly with time. After a second differentiation with respect to time, we get

$$\ddot{L} = g$$

This means that the acceleration of the elevator is constant. The elevator is uniformly accelerated upward. The equations connecting the primed and unprimed coordinates are different from equations (4). The transformation for the vertical coordinates is now

$$z' = z - \frac{1}{2}gt^2 \tag{8}$$

The other equations in equations (4) don't change:

$$t' = t$$
$$x' = x$$
$$y' = y$$

These four equations are our new coordinate transformation to represent the relationship between coordinates that are accelerated relative to each other.

We will continue to assume that in the z, or unprimed, coordinate system, the laws of physics are exactly what Newton taught us. In other words, the stationary reference frame is inertial, and we have $F = m\ddot{z}$. But the primed frame is no longer inertial. It is in uniform acceleration relative to the unprimed frame. Let's ask what the laws of physics are now in the primed frame of reference. We have to do the operation of differentiating twice over again on equation (8). We know the answer:

$$\ddot{z}' = \ddot{z} - g \tag{9}$$

Ah ha! Now the primed acceleration and the unprimed acceleration differ by an amount g. To write Newton's equations in the primed frame of reference, we multiply both sides of equation (9) by m, the particle mass, and we replace $m\ddot{z}$ by F. We get

$$m\ddot{z}' = F - mg \tag{10}$$

We have arrived at what we wanted. Equation (10) looks like a Newton equation, that is, mass times acceleration is equal to some term. That term, $F - mg$, we call the force in the primed frame of reference. You notice, as expected, that the force in the primed frame of reference has an extra term: the mass of the particle times the acceleration of the elevator, with a minus sign.

What is interesting about the "fictitious force" $-mg$, in equation (10), is that it looks exactly like the force exerted on the particle by gravity on the surface of the Earth or the surface of any kind of large massive body. That is why we called the acceleration g. The letter g stood for gravity. It looks like a uniform gravitational field. Let me spell out in what sense it looks like gravity. The special feature of gravity is that gravitational forces are proportional to mass – the same mass that appears in Newton's equation of motion. We sometimes say that *the gravitational mass is the same as the inertial mass.* That has deep implications. If the equation of motion is

$$F = ma \tag{11}$$

and the force itself is proportional to mass, then the mass cancels in equation (11). That is a characteristic of gravitational forces: for a small object moving in a gravitational force field, its motion doesn't depend on its mass. An example is the motion of the Earth about the Sun. It is independent of the mass of the Earth. If you know where the Earth is at time t, and you know its velocity at that time, then you can predict its trajectory. You don't need to know what the Earth's mass is.

Equation (10) is an example of *fictitious force* – if you want to call it that – mimicking the effect of gravity. Most people before Einstein considered this largely an accident. They certainly knew that the effect of acceleration mimics the effect of gravity, but they didn't pay much attention to it. It was Einstein who said: look, this is a deep principle of nature that gravitational forces cannot be distinguished from the effect of an accelerated reference frame.

If you are in an elevator without windows and you feel that your body has some weight, you cannot say whether the elevator, with you inside, is resting on the surface of a planet or, far away from any massive body in the universe, some impish devil is accelerating your elevator. That is the *equivalence principle*. It extends the relativity principle, which said you can juggle in the same way at rest or in a railway car in uniform motion. With a simple example, we have equated accelerated motion and gravity. We have begun to explain what is meant by the sentence: "gravity is in some sense the same thing as acceleration."

We have to discuss this result a bit, though. Do we really believe it totally or does it have to be qualified? Before we do that, let's draw some pictures of what these various coordinate transformations look like.

Curvilinear Coordinate Transformations

Let's first consider the case where $L(t)$ is proportional to t. That is when we have

$$z' = z - vt$$

In figure 3, every point – also called *event* – in space-time has a pair of coordinates z and t in the stationary frame and also a pair of coordinates z' and t' in the elevator frame. Of course, $t' = t$ and we left out the two other spatial coordinates x and y, which don't change between the stationary frame and the elevator. We represented the time trajectories of fixed z with dotted lines and of fixed z' with solid lines.

A fundamental idea to grasp is that events in space-time exist irrespective of their coordinates, just as points in space don't depend on the map we use. Coordinates are just some sort of convenient *tags*. We can use whichever we like. We'll stress it again after we have looked at figures 3 and 4.

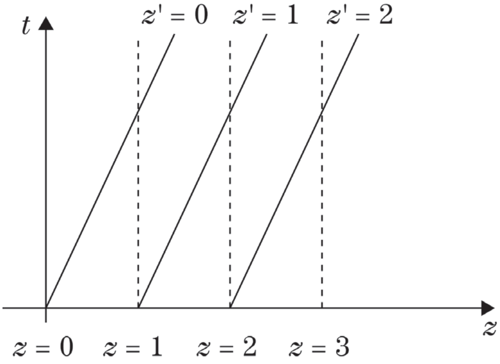

Figure 3: Linear coordinate transformation. The coordinates $(z',\ t')$ are represented in the basic coordinates $(z,\ t)$. An event is a point on the page. It has one set of coordinates in the $(z,\ t)$ frame and another set in the $(z',\ t')$ frame. Here the transformation is simple and linear.

That is called a *linear coordinate transformation* between the two frames of reference. Straight lines go to straight lines, not surprisingly since Newton tells us that free particles move in straight lines in an inertial frame of reference. What is a straight line in one frame had therefore better be a straight line in the other frame. Not only do free particles move in straight lines in space, when we add x and y, but their trajectories are straight lines in space-time – straight in space and with uniform velocity.

Let's do the same thing for the accelerated coordinate system. The transformation equation is now equation (8) linking z' and z. The other coordinates don't change. Again, in figure 4, every point in space-time has two pairs of coordinates $(z,\ t)$ and $(z',\ t')$. The time trajectories of fixed z, represented with dotted lines, don't change. But now the time trajectories of fixed z' are parabolas lying on their side. We can even represent negative times in the past. Think of the elevator that was initially moving downward with a negative velocity but a positive acceleration g (in other words, slowing down). Then the elevator bounces back upward with the same acceleration g. Each parabola is just shifted relative to the previous one by one unit to the right.

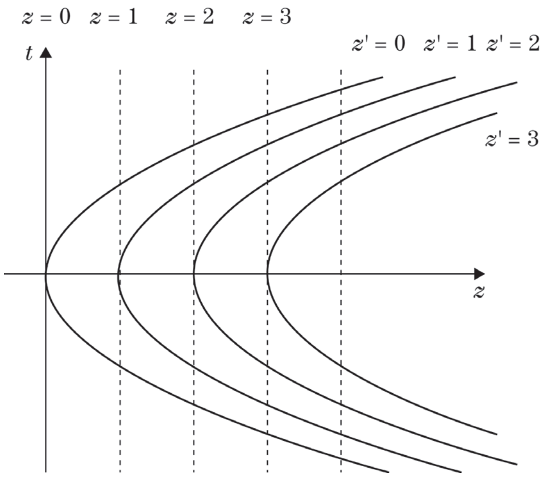

Figure 4: Curvilinear coordinate transformation.

What figure 4 illustrates is, not surprisingly, that straight lines in one frame are not straight lines in the other frame. They become curved lines. As regards the lines of fixed t or fixed t', they are of course the same horizontal straight lines in both frames. We haven't represented them.

We should view figure 4 as just two sets of coordinates to locate each point in space-time. One set of coordinates has straight axes, while the second – represented in the first frame – is curvilinear. Its lines z' = constant are actually curves, while its lines t' = constant are horizontal straight lines. So it is a *curvilinear coordinate transformation.*

Let's insist on the way to interpret and use figure 4 because it is fundamental to understand it very well if we want to understand the theory of relativity – special relativity and even more importantly general relativity. The page represents space-time – here, one spatial dimension and one temporal dimension.

Points (= events) in space-time are points on the page. An event *does not have two positions* on the page, i.e., in space-time. *It has* *only one position* on the page. But this position can be located, mapped, "charted" one also says, using several different systems of reference. A system of reference, also called a frame of reference, is nothing more than a complete set of "labels," if you will, attaching one label (consisting of two numbers, because our space-time here is two-dimensional) to each point, i.e., to each event.

In a two-dimensional space, the system of reference can be geometrically simple, like orthogonal Cartesian axes in the plane. However this is not a necessity. For one thing, on Earth, which is not a plane, the axes are not straight lines. The usual axes used by cartographers and mariners are meridians and parallels. But on a 2D surface, be it a plane or not, we can imagine quite fancy or intricate curvilinear lines to serve as a frame of reference – so long as it attaches unequivocally two numbers to each (by definition, fixed) point. This is what figure 4 does in the space-time made of one temporal and one spatial dimension represented on the page. We will see many more in lecture 2.

Something Einstein understood very early is this:

There is a connection between gravity and curvilinear coordinate transformations of space-time.

Special relativity was only about linear transformations – transformations that take uniform velocity to uniform velocity. Lorentz transformations are of that nature. They take straight lines in space-time to straight lines in space-time. However, if we want to mock up gravitational fields with the effect of acceleration, we are really talking about transformations of coordinates of space-time that are curvilinear. That sounds extremely trivial. When Einstein said it, probably every physicist knew it and thought: "Oh yeah, no big deal." But Einstein was very clever and very persistent. He realized that if he followed very far the consequences of this, he could then answer questions that nobody knew how to answer.

Let's look at a simple example of a question that Einstein answered using the curved coordinates of space-time representing acceleration, and consequently, if the two are the same, gravity. The question is: what is the influence of gravity on light?

Effect of Gravity on Light

When Einstein first asked himself the question "what is the influence of gravity on light"? around 1907, most physicists would have answered: "There is no effect of gravity on light. Light is light. Gravity is gravity. A light wave moving near a massive object moves in a straight line. It is a law of light that it moves in straight lines. And there is no reason to think that gravity has any effect on it."

But Einstein said: "No, if this equivalence principle between acceleration and gravity is true, then gravity must affect light. Why? Because acceleration affects light." It was again one of these arguments that you could explain to a clever child.

Let's imagine that, at $t = 0$, a flashlight (today we might use a laser pointer) emits a pulse of light in a horizontal direction from the left side of the elevator; see figure 5. The light then travels across to the right side with the usual speed of light c. Since the stationary frame is assumed to be an inertial frame, the light moves in a straight line in the stationary frame.

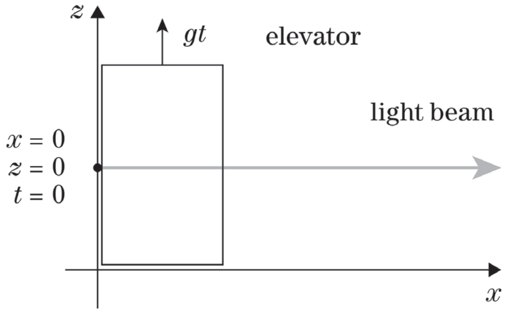

Figure 5: Trajectory of a light beam in the *stationary* reference frame.

The equations for the light ray are

$$\begin{aligned} x &= ct \\ z &= 0 \end{aligned} \tag{12}$$

The first of these equations just says that the light moves across the elevator with the speed of light – no surprise here.

The second says that in the stationary frame the trajectory of the light beam is horizontal.

Let's express the same equations in terms of the primed coordinates. The first equation becomes

$$x' = ct$$

And the second takes the more interesting form

$$z' = -\frac{g}{2}t^2$$

It says that as the light ray moves across the elevator, at the same time the light ray accelerates downward – toward the floor – just as if gravity were pulling it.

We can even eliminate t from the two equations and get an equation for the curved trajectory of the light ray:

$$z' = -\frac{g}{2c^2}x'^2 \tag{13}$$

Thus, the trajectory, in the primed frame of reference, is a parabola, not a straight line.

But, said Einstein, if the effect of acceleration is to bend the trajectory of a light ray, then so must be the effect of gravity.

Andy: *Gee Lenny, that's really simple. Is that all there is to it?*

Lenny: *Yup Andy, that's all there is to it. And you can bet that a lot of physicists were kicking themselves for not thinking of it.*

To summarize, in the stationary frame, the photon trajectory (figure 5) is a straight line, while in the elevator reference frame, it is a parabola (figure 6).

Let's imagine three people arguing. I am in the elevator, and I say: "Gravity is pulling the light beam down." You are in the stationary frame, and you say: "No, it's just that the elevator is accelerating upward; that makes it look like the light beam moves on a curved trajectory." And Einstein says: "They are the same thing!"

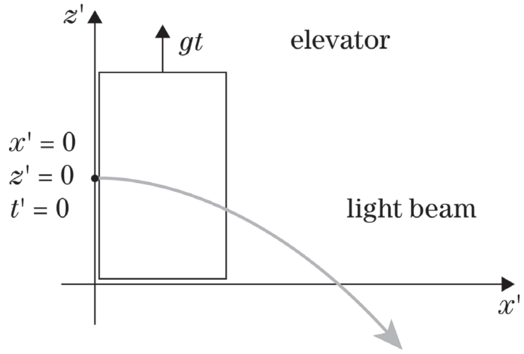

Figure 6: Trajectory of a light beam in the *elevator* reference frame.

This proved to him that a gravitational field must bend a light ray. As far as I know, no other physicist understood this at the time.

In conclusion, we have learned that it is useful to think about curvilinear coordinate transformations in space-time.

When we do think about curvilinear coordinates transformations, *the form of Newton's laws changes.* One of the things that happen is that apparent gravitational fields materialize, which are physically indistinguishable from ordinary gravitational fields.

Well, are they really physically indistinguishable? For some purposes yes, but not for all. So let's turn now to real gravitational fields, namely gravitational fields of gravitating objects like the Sun or the Earth.

Tidal Forces

Figure 7 represents the Earth, or the Sun, or any massive body. The gravitational acceleration doesn't point vertically on the page. It points toward the center of the body.

It is pretty obvious that there is no way that you could do a coordinate transformation like we did in the preceding section that would remove the effect of the gravitational field. Yet, if you are in a small laboratory in space and that laboratory is allowed to simply fall toward Earth, or toward whatever massive object you are considering, then you will think that in that laboratory there is no gravitational field.

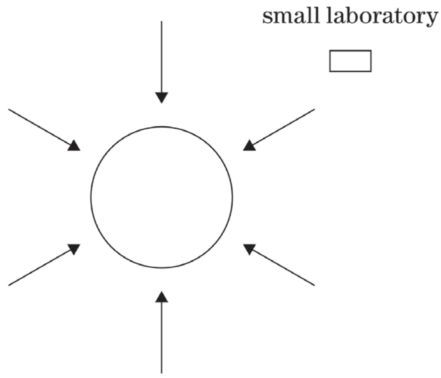

Figure 7: Gravitational field of a massive object, and small laboratory falling toward the object, experiencing inside itself no gravitation.

Exercise 1: If we are falling freely in a uniform gravitational field, prove that we feel no gravity and that things float around us like in the International Space Station.

But, again, there is no way *globally* to introduce a coordinate transformation that is going to get rid of the fact that there is a gravitational field pointing toward the center. For instance, a very simple transformation similar to equations (12) might get rid of the gravity in a small portion on one side of the Earth, but the same transformation will increase the gravitational field on the other side. Even more complex transformations would not solve the problem.

One way to understand why we can't get rid of gravity is to think of an object that is not small compared to the gravitational field. My favorite example is a 2000-mile man who is falling in the Earth's gravitational field; see figure 8. Because he is so big, different parts of his body feel different gravitational fields. Remember that the farther away you are, the weaker is the gravitational field.

His head feels a weaker gravity than his feet. His feet are being pulled harder than his head. He feels like he is being stretched, and that stretching sensation tells him that there is a gravitating object nearby. The sense of discomfort that he feels, due to the nonuniform gravitational field, cannot be removed by switching to a free-falling reference frame. Indeed, no change of mathematical description whatsoever can change this physical phenomenon.

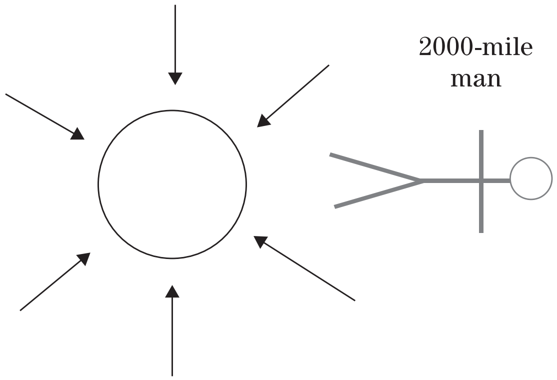

Figure 8: A 2000-mile man falling toward Earth.

The forces he feels are called *tidal forces*, because they play an important role in the phenomenon of tides, too. They cannot be removed by a coordinate transformation. Let's also see what happens if he is falling not vertically but sideways, staying perpendicular to a radius. In that case his head and his feet will be at the same distance from Earth. Both will be subjected to the same force in magnitude pointing to Earth. But since the force directions are radial, they are not parallel. The force on his head and the force on his feet will both have a component along his body. A moment's thought will convince us that the tidal forces will compress him, his feet and head being pushed toward each other. This sense of compression is again not something that we can remove by a coordinate transformation. Being stretched or shrunk, or both, by the Earth's gravitational field – if you are big enough – is an invariant fact.

In summary, it is not quite true that gravity is equivalent to going to an accelerated reference frame.

Andy: *Aha! So Einstein was wrong after all.*

Lenny: *Well, Einstein was wrong at times, but no, Andy, this was not one of those times. He just had to qualify his statement and make it a bit more precise.*

What Einstein really meant was that small objects, for a small length of time, cannot tell the difference between a gravitational field and an accelerated frame of reference.

It raises the following question: if I present you with a force field, does there exist a coordinate transformation that will make it vanish? For example, the force field inside the elevator, associated with its uniform acceleration with respect to an inertial reference frame, was just a vertical force field pointing downward and uniform everywhere. There was a transformation canceling it: simply use z- instead of z'-coordinates. It is a nonlinear coordinate transformation. Nevertheless, it gets rid of the force field.

With other kinds of coordinate transformations, you can make the gravitational field look more complicated, for example transformations that affect also the x-coordinate. They can make the gravitational field bend toward the x-axis. You might simultaneously accelerate along the z-axis while oscillating back and forth on the x-axis. What kind of gravitational field do you see? A very complicated one: it has a vertical component and it has a time-dependent oscillating component along the x-axis.

If instead of the elevator you use a merry-go-round, that is, a carousel, and instead of the $(x',\ z',\ t)$ coordinates of the elevator, you use polar coordinates $(r,\ \theta,\ t)$, an object that in the stationary frame was fixed, or had a simple motion like the light beam, may have a weird motion in the frame moving with the merry-go-round. You may think that you have discovered some repulsive gravitational field phenomenon. But no matter what, the reverse coordinate change will reveal that your apparently messy field is only the consequence of a coordinate change. By choosing funny coordinate transformations, you can create some pretty complicated fictitious, apparent, also called *effective*, gravitational fields. Nonetheless they are not genuine, in the sense that they don't result from the presence of massive objects.

If I give you the field everywhere, how do you determine whether it is fictitious or genuine, i.e., whether it is just the sort of fake gravitational field resulting from a coordinate transformation to a frame with all kinds of accelerations with respect to a simple inertial one, or it is a real gravitational field?

If we are talking about Newtonian gravity, there is an easy way. You just calculate the tidal forces. You determine whether that gravitational field will have an effect on an object that will cause it to squeeze and stretch. If calculations are not practical, you take an object, a mass, a crystal. You let it fall freely and see whether there were stresses and strains on it. If the crystal is big enough, these will be detectable phenomena. If such stresses and strains are detected, then it is a real gravitational field as opposed to only a fictitious one.

On the other hand, if you discover that the gravitational field has no such effect, that any object, wherever it is located and let freely to move, experiences no tidal force – in other words, that the field has no tendency to distort a free-falling system – then it is a field that can be eliminated by a coordinate transformation.

Einstein asked himself the question: what kind of mathematics goes into trying to answer the question of whether a field is a genuine gravitational one or not?

Non-Euclidean Geometry

After his work on special relativity, and after learning of the mathematical structure in which Minkowski[7] had recast it, Einstein knew that special relativity had a geometry associated with it. So let's take a brief rest from gravity to remind ourselves of this important idea in special relativity. Special relativity was the main subject of the third volume of TTM. Here, however, the only thing we are going to use about special relativity is that space-time has a geometry.

[7] Hermann Minkowski (1864–1909), Polish-German mathematician and theoretical physicist.

In the Minkowski geometry of special relativity, there exists a kind of distance between two points, that is, between two events in space-time; see figure 9.

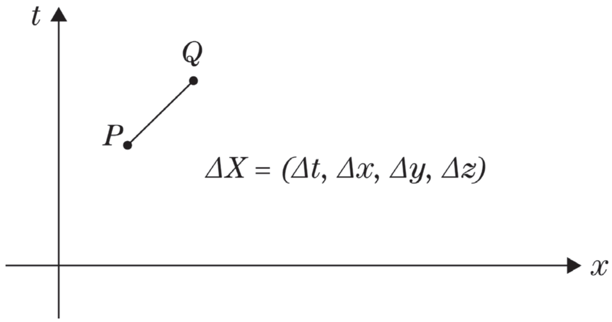

Figure 9: Minkowski geometry: a 4-vector going from P to Q.

The distance between P and Q is not the usual Euclidean distance that we could be tempted to think of. It is defined as follows. Let's call ΔX the 4-vector going from P to Q. To the pair of points P and Q we assign a quantity denoted $\Delta\tau$, defined by

$$\Delta\tau^2 = \Delta t^2 - \Delta x^2 - \Delta y^2 - \Delta z^2$$

Notice that $\Delta\tau$ does not satisfy the usual properties of a distance. In particular, $\Delta\tau^2$ can be positive or negative; and it can be zero for two events that are not identical. The reader is referred to volume 3 of TTM for details. Here we only give a brief refresher.

The quantity $\Delta\tau$ is called the *proper time* between P and Q. It is an invariant under Lorentz transformations. That is why it qualifies as a sort of distance, just as in three-dimensional (3D) Euclidean space the distance between two points, $\Delta x^2 + \Delta y^2 + \Delta z^2$, is invariant under isometries.

We also define a quantity Δs by

$$\Delta s^2 = -\Delta t^2 + \Delta x^2 + \Delta y^2 + \Delta z^2$$

We call Δs the *proper distance* between P and Q. Of course, $\Delta\tau$ and Δs are not two different concepts. They are the same – just differing by an imaginary factor i. They are just two ways to talk about the Minkowski "distance" between P and Q. Depending on which physicist is writing the equations, they will rather use $\Delta\tau$ or Δs as the distance between P and Q.

Einstein knew about this non-Euclidean geometry of special relativity. In his work to include gravity, and to investigate the consequences of the equivalence principle, he also realized that the question we asked at the end of the previous section – are there coordinate transformations that can remove the effect of forces? – was very similar to a certain mathematics problem that had been studied at great length by Riemann. It is the question of deciding whether a geometry is flat or not.

Riemannian Geometry

What is a flat geometry? Intuitively, it is the following idea: the geometry of a page is flat. The geometry of the surface of a sphere or a section of a sphere is not flat. The *intrinsic geometry* of the page remains flat even if we furl the page like in figure 10. We will expound mathematically on the idea in a moment.

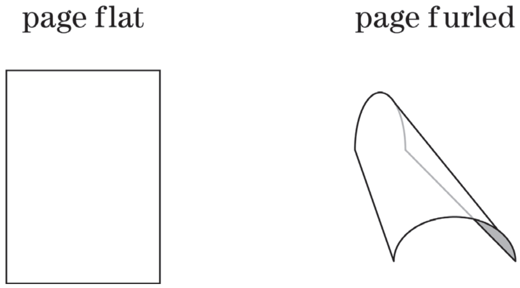

Figure 10: The intrinsic geometry of a page remains flat.

For now, let's just say that the intrinsic geometry of a surface is the geometry that a two-dimensional bug roaming on it, equipped with tiny surveying tools, would see if it were trying to establish an ordnance survey map of the surface.

If the bug worked carefully, it might see hills and valleys, bumps and troughs, if there were any, but it would not notice that the page is furled. We see it because *for us* the page is embedded in the 3D Euclidean space we live in. By unfurling the page, we can make its flatness obvious again.

Einstein realized that there was a great deal of similarity in the two questions of whether a geometry is non-flat and whether a space-time has a real gravitational field in it. Riemann had studied the first question. But Riemann had never dreamt about geometries that have a minus sign in the definition of the square of the distance. He was thinking about geometries that were non-Euclidean but were similar to Euclidean geometry – not Minkowski geometry.

Let's start with the mathematics of Riemannian geometry, that is, of spaces where the distance between two points may not be the Euclidean distance, but in which the square of the distance is always positive.[8]

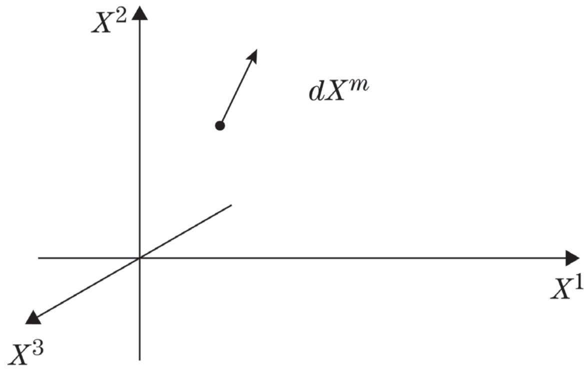

Figure 11: Small displacement between two points in a space.

We look at two points in a space; see figure 11. In our example there are three dimensions, therefore three axes, X^1, X^2, and X^3. There could be more. Thus a point has three coordinates, which we can write as X^m, where m is understood to run from 1 to 3 or to whatever number of axes there is. And a little shift between one point and another nearby has three components, which can be denoted ΔX^m or, if it is to become an infinitesimal, dX^m.

[8]In mathematics, they are called *positive definite distances.*

If this space has the usual Euclidean geometry, the square of the length of dX^m is given by Pythagoras theorem

$$dS^2 = (dX^1)^2 + (dX^2)^2 + (dX^3)^2 + \dots \tag{14}$$

If we are in three dimensions, then there are three terms in the sum. If we are in two dimensions, there are two terms. If the space is 26-dimensional, there are 26 of them and so forth. That is the formula for Euclidean distance between two points in Euclidean space.

For simplicity and ease of visualization, let's focus on a two-dimensional space. It can be the ordinary plane, or it can be a two-dimensional surface that we may visualize embedded in 3D Euclidean space, as in figure 12.

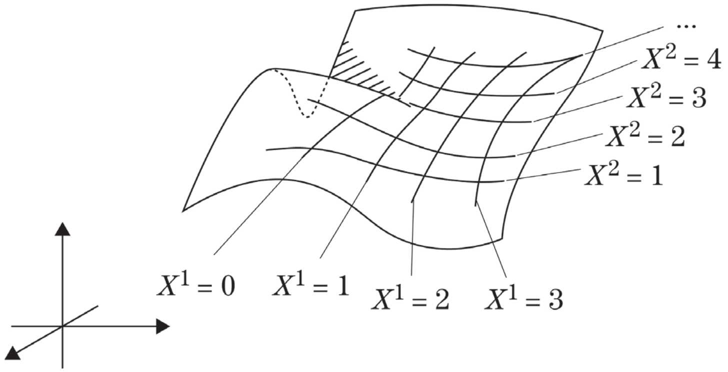

Figure 12: Two-dimensional manifold (i.e., 2D surface) and its curvilinear coordinates viewed embedded in ordinary 3D euclidean space.

There is nothing special about two dimensions for such a surface, except that it is easy to visualize. Mathematicians think of "surfaces" even when they have more dimensions. Usually they don't call them surfaces but *manifolds* or sometimes *varieties*.

Gauss had already understood that on curved surfaces the formula for the distance between two points was more complicated in general than equation (14). Indeed, we must not be confused by the fact that in figure 12 the surface is shown embedded in the usual three-dimensional Euclidean space. This is just for convenience